Logical Omniscience and Inconsistent Belief [1]

Jan O.M. Jaspars

Institute for Language Technology & AI
Tilburg University
PO Box 90153
5000 LE TILBURG
THE NETHERLANDS
EMAIL: jaspars@kub.nl

Abstract

In this paper we investigate a modal logic which has been set up by a Kripke semantics in which accessible worlds can cluster, inspired by the so-called fusion semantics of Rescher and Brandom [ReB]. This modal logic is motivated from an epistemological viewpoint. A cognitive agent is free to confuse his "doxastic alternatives" (accessible worlds). Such an agent is then said to believe a proposition whenever it is verified by some world in every accessible fused set. This causes the underlying logic to be abnormal (weaker than **K**). In terms of epistemic logic, we have a partial disappearance of logical omniscience, and moreover we are able to deal with inconsistent beliefs, such that an agent does not have to believe everything whenever he is confronted with mutually contradictory information. From a technical point of view the fused Kripke semantics stands relatively close to the ordinary Kripke semantics among other abnormal modal logics[2].

1 Introduction

Logical omniscience is the problem that we inherit from possible worlds analysis of cognitive propositional attitudes, such as knowledge and belief. The knowledge or belief of a cognitive agent is interpreted as being inversely proportional to its uncertainty. Possible worlds *accessible* to the agent define this uncertainty, since every such world might be the real one. What an agent knows or believes is then determined by the information that is verified by all these accessible worlds (uncertainties).

The problem of logical omniscience arises by the strict logical behavior of these worlds [Ran] [Hinb]. This means that our agent has to know or belief all the logical consequences of its knowledge or belief, which seems to be much too idealistic. This problem is related to the representation of inconsistent belief, because whenever an

[1]This research is supported by the programme 'Dialogue management and knowledge acquisition' (DenK) of the Tilburg-Eindhoven Organization for Inter-University Cooperation (SOBU).

[2]The similarity of fused and normal Kripke semantics for modal logic entails easy meta-theoretical proofs, which can be adopted from standard modal logic. For a more technical elaboration on fused semantics see [Jas].

agent has inconsistent belief, it has to belief everything, due to his logical omniscience.[3] This phenomenon is an extremely unnatural consequence of logical omniscience.

Most often these two problems are solved by liberalizing the logical behavior of possible worlds in two directions. Firstly, a possible world can leave a proposition undefined. Such a world has an incomplete or partial character with respect to this proposition. Secondly, a world can be logically impossible in the sense that it overdefines a certain proposition. It recognizes such a proposition as both true and false.

In this paper we will focus on the impossible character that worlds may have as doxastic alternatives. Here we will only concern ourselves with representing inconsistent belief and its consequences with respect to the logical omniscient capacities of a cognitive agent. According to us the fourth truth-value (*both* = both true and false) is not necessary for representing inconsistent belief states. We propose an alternative analysis for overdefinedness, using fusion of classical worlds, of which the basic ideas were originally introduced by Rescher and Brandom [ReB]. In the set up presented in this paper an agent may confuse its accessible worlds.[4]

1.1 Belnap's machine

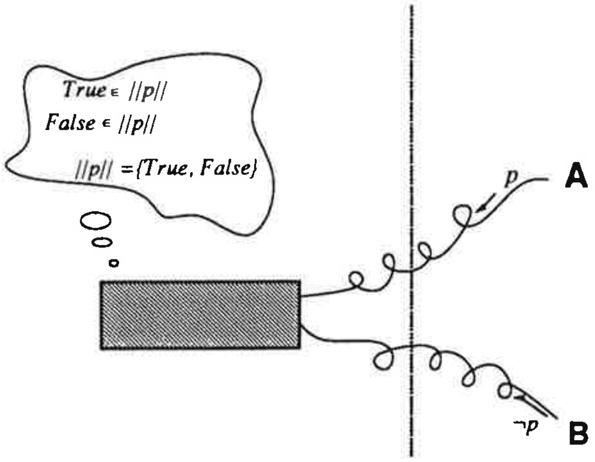
The four valued approach towards inconsistent beliefs started with Belnap's article on machines that have to deal with inconsistent information [Bel]. He motivated his four valued logic by a machine dialogue. Suppose we have a naive machine that cannot distinguish different users, let alone weigh the information by assigning reliabilities to the users. Belnap was concerned about how such a machine would have to reason without believing everything whenever it would be confronted with contradictory information.

Consider the following configuration.

---

[3]Possible world models for inconsistent belief are these models in which there are no possible worlds accessible to the agent.

[4]Rescher and Brandom motivated their semantics with fused worlds from an ontological viewpoint. They only introduced a wider semantics. For possible world semantics for modal logics they gave a transformation for ordinary possible world models to fused possible world models, such that the classical normal minimal modal logic is still valid, and so logical omniscience is still there. In this paper we will look at fused possible world models with a free accessibility-relation. Vardi [V86] already used Rescher and Brandom's approach for doxastic logics. There is only one big difference with our approach. In Vardi's models an agent confuses all his doxastic alternatives. This is what he calls *local reasoning*. We will look at possible world models, in which the degree of (con)fusion is arbitrary.



One user, A, tells the machine that p is the case, while another user, B, tells him the contrary: $\neg p$. If a user C were to consult Belnap's machine on p, it would respond: "p is both true and false".

There are a few pragmatic problems related to this analysis. A and B's utterances on p are taken to be incomplete by Belnap's machine. A's utterance is interpreted as "***true*** is one (element [5]) of the truth values of p". Successively, B's contradictory addition disambiguates the interpretation of p in a four valued interpretation. Such a dialogue strategy is illegal according to two of the conversational maxims of Grice [Gri]. Firstly, the machine interpreted A's utterance as if A and B were not sure of the content of their messages. Furthermore, the machine acts as if A and B are withholding information. By the first argument Belnap's machine violates the maxim of *quality*, and by the second argument the maxim of *quantity*.

Consistent information would be taken to be incomplete. If B did not contradict A, and if C would afterwards consult the machine on p, it would give him the ridiculous answer: "p is either only true or both true and false". Maybe there are ways to get around this, but Belnap does not give us the techniques.

Besides these obligations, four valued logic does not safeguard us from inconsistencies. If A were saying that p is only true – an addition that is not necessary in Gricean dialogues – it would still contradict B's message. In order to deal with such a situation, following Belnap's argumentation, we would have to equip our machine with sixteen truth values. [6]

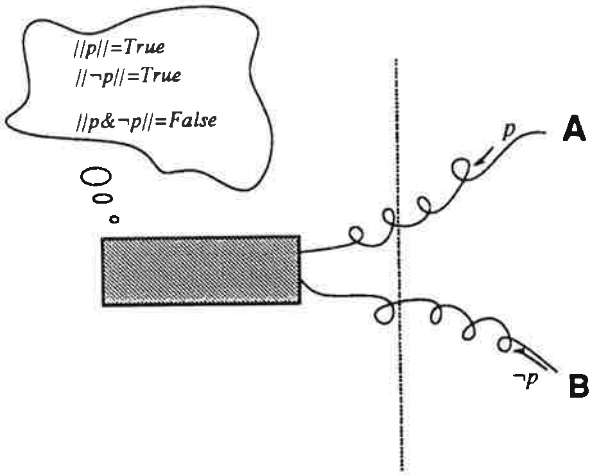
1.2 Rescher and Brandom's machine

A much better idea of dealing with inconsistent information is to interpret such information as fusion of consistent parts (classical worlds), such as was introduced by Rescher and Brandom [ReB]. In a possible world framework for logics of belief this must be

[5] In four valued logic truth-values are taken to be subsets of the classical truth values. $\emptyset$ is the truth value undefined. $\{\mathit{true}\}$, $\{\mathit{false}\}$, $\{\mathit{true}, \mathit{false}\}$ stand for "only true", "only false" and "both true and false" respectively.

[6] Following this line would make any $2^{(2^{(\cdot\cdot 2 \cdot\cdot)})}$ valued logic as useful as four valued logic. One may argue that A and B may only be two-valued reasoners. This unfair play would have to be paid off if messages come from other Belnap machines.

interpreted as an agent that confuses his uncertainties or accessible worlds.[7] Different worlds are taken to be identical by this agent. In the case of Belnap's dialogue situation, the machine (con)fuses the users A and B. But it would not conclude $p \wedge \neg p$ because no one told him so. The following figure illustrates the thought of a machine which would think in the fashion of Rescher and Brandom.



The machine believes all the messages that it gets from the fused group of users. It takes the users as its information sources, and would believe anything whenever it is verified by at least one of these. [8]

In this paper we will take Rescher and Brandom's fusion as the source of inconsistencies. We will construct a modal logic, in which a belief-operator gets explicit status in the logic, such that it is possible to have inconsistent beliefs without concluding the absurd ($\perp$). Contrary to four valued logic [9] we will not withhold the agent from concluding everything from $\perp$, but rather stop him from joining inconsistent information. In general we do not have the following derivation, which is valid in normal modal logic.

$$\Box p \wedge \Box \neg p \Longrightarrow \Box(p \wedge \neg p) \Longrightarrow \Box \varphi$$ [10]

In our fused modal logic the first implication is no longer valid. In four valued logics the second consequence is eliminated.

[7] Lots of human inconsistent beliefs seem due to confusion of time-points.

[8] Such an avoidance of concluding the absurd proposition $\perp$, can be compared by interpretation of large religious works as the bible. A believer believes everything that the prophets and evangelists tell him. However it does not necessarily have to believe propositions that would follow from joining information from different messagers. This differs from political reasoning, where four valued logic seems to have large popularity. "Yes and no" is not only an answer that makes sense in political debate, but it is often taken to be mysteriously interesting.

[9] See [Lev] for a modal logic using four valued logic.

[10] In Rescher and Brandom's terminology one does not believe self-inconsistent information.

2 Fused modal logic

To start with we define the language that we will use. The language $\mathcal{L}^{\Box}$ is just the propositional language together with a modal operator $\Box$, which symbolizes a single agent's belief.

Definition 1

$\mathcal{L}^{\Box}$ is the smallest set such that

$I\!P \subset \mathcal{L}^{\Box}$, where $I\!P$ is a finite non-empty set of primitive propositions.

$\varphi \in \mathcal{L}^{\Box} \Longrightarrow \neg\varphi \in \mathcal{L}^{\Box}$

$\varphi \in \mathcal{L}^{\Box},\ \psi \in \mathcal{L}^{\Box} \Longrightarrow (\varphi \wedge \psi) \in \mathcal{L}^{\Box}$

$\varphi \in \mathcal{L}^{\Box} \Longrightarrow \Box\varphi \in \mathcal{L}^{\Box}$

Furthermore we use well-known abbreviations such as $\top, \bot, \vee, \rightarrow$ and $\Diamond$ ($\top := p\vee\neg p, \bot := \neg\top, \varphi\vee\psi := \neg(\neg\varphi\wedge\neg\psi), \varphi \rightarrow \psi := (\neg\varphi\vee\psi), \Diamond\varphi := \neg\Box\neg\varphi$). $\Box\varphi$ should be interpreted as 'the agent believes that φ'.

The fused Kripke semantics has the following straightforward format.

Definition 2

A fused Kripke-frame is a pair $\langle W, R\rangle$ consisting of a non-empty set of worlds W, and an accessibility-relation $R \subseteq W \times \wp(W)/\{\emptyset\}$. R links worlds to non-empty sets of (fused) worlds.

A fused Kripke-model is a triple $\langle W, R, V\rangle$ such that $\langle W, R\rangle$ is a fused Kripke-frame, and V a local valuation: $V : W \times I\!P \longrightarrow \{0, 1\}$.

Ordinary Kripke-models with accessibility relations in $W \times W$, can be understood as a best case of this fused semantics. The Rescher and Brandom-like thinking machine in the Belnap dialogue configuration is then a worst case.

Truth-conditional semantics is realized by the following composition:

Definition 3

Let $M = \langle W, R, V\rangle$ be a fused Kripke-model, and let $w \in W$.

$M, w \models p \Leftrightarrow V(w, p) = 1$ *for all* $p \in I\!P$

$M, w \models \neg\varphi \Leftrightarrow M, w \not\models \varphi$

$M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi\ \&\ M, w \models \psi$

$M, w \models \Box\varphi \Leftrightarrow \forall W' \subseteq W : (wRW' \Rightarrow (\exists w' \in W' : M, w' \models \varphi))$

$\Phi \models \varphi$ for $\Phi \subseteq \mathcal{L}^{\Box}$ means that every world in any model that verify all formulae in Φ also verifies φ. $\models \varphi$ means that φ holds in all worlds of any model.

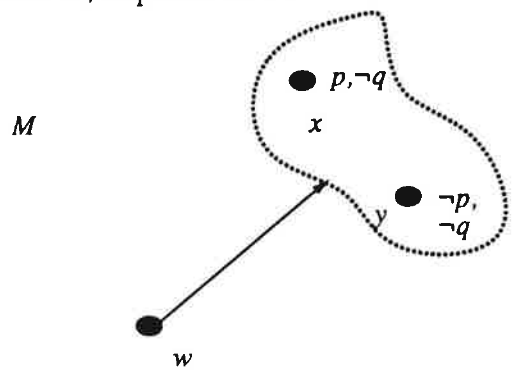
We will use single lower case letters as denotations for single worlds (v, w, x, y, z), while capital letters denote non-empty sets of worlds (V, W, X, Y, Z).

The last clause of the truth-conditions states that for every accessible set of worlds, there must be at least one element that verifies φ.

Observation 1

$$\models \varphi \Rightarrow \models \Box\varphi$$
$$\models \varphi \rightarrow \psi \Rightarrow \models \Box\varphi \rightarrow \Box\psi$$
$$\not\models \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

A counter-model of the last formula, the so-called K-axiom (or distribution-schema), is given by the simple model M, depicted below.



$M, y \models p \rightarrow q$, and so $M, w \models \Box(p \rightarrow q)$

$M, x \models p$, therefore $M, w \models \Box p$

$M, x \not\models q$ and $M, y \not\models q$, and so $M, w \not\models \Box q$

This falsification of the so-called **K**-axiom, deminishes the logical omniscient capacities. Beliefs are no longer closed under logical consequence. Of course some parts of logical omniscience remain. For example we still have necessitation: one always believes all tautologies. The fused model depicted above is also a counter-model for $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$. Simply substitute p for φ, and $\neg p$ for ψ.

Definition 4

$\mathbf{RB}$[11] is the minimal fused modal logic. In $\mathbf{RB}$ there is one global axiom, stating that all propositional tautologies are $\mathbf{RB}$-theorems.

A. $\Phi \vdash_{PC} \varphi \Rightarrow \Phi \vdash_{RB} \varphi$[12]

Besides this global axiom, there are three inference rules.

[11] After Rescher and Brandom.

[12] **PC** is the classical propositional calculus.

R1. $\vdash_{RB} \varphi \Rightarrow \vdash_{RB} \Box\varphi$ *(Necessitation or Generalization)*

R2. $\varphi, \varphi \rightarrow \psi \vdash_{RB} \psi$ *(Modus Ponens)*

R3. $\vdash_{RB} \varphi \rightarrow \psi \Rightarrow \vdash_{RB} \Box\varphi \rightarrow \Box\psi$

This logic is complete and sound with respect to the class of fused Kripke-models. Soundness of **RB** has already been checked by observation 1 for the main part. Note that all worlds in the fused models are classical, and therefore the propositional calculus is part of **RB**. The completeness can be proved in Henkin style. We build a model of maximal **RB**-consistent sets: the fused canonical model. This model turns out to be a countermodel for every formula which is not an **RB**-theorem.

2.1 The canonical model

At first we define the notion of maximal **RB**-consistent sets.

Definition 5

A set $\Lambda \subseteq \mathcal{L}^\Box$ *is said to be* **RB**-*consistent if for every sequence* $\{\alpha_i\}_{i=1}^m \subseteq \Lambda$:

$$\nvdash_{RB} \neg(\alpha_1 \wedge .. \wedge \alpha_m)$$

A set $\Gamma \subseteq \mathcal{L}^\Box$ *is said to be a maximal* **RB**-*consistent set if it is* **RB**-*consistent, and it is maximal in the sense that it has no proper* **RB**-*consistent extension.*

$$\forall \Delta \supset \Gamma : \Delta \textit{ is not } \mathbf{RB}\textit{-consistent.}$$

Observation 2

Let Γ be a maximal **RB**-consistent set.

- $\Gamma \vdash_{RB} \varphi \Rightarrow \varphi \in \Gamma$. In particular, all **RB**-theorems are in Γ.
- $\varphi \notin \Gamma \Leftrightarrow \neg\varphi \in \Gamma$
- $\varphi \wedge \psi \in \Gamma \Leftrightarrow \varphi \in \Gamma$ and $\psi \in \Gamma$.
- $\varphi \vee \psi \in \Gamma \Leftrightarrow \varphi \in \Gamma$ or $\psi \in \Gamma$.

By these maximal **RB**-consistent sets we construct the canonical model of **RB**.

Definition 6

The canonical model of **RB** *is the triple* $M_{RB} = \langle W_{RB}, R_{RB}, V_{RB} \rangle$ *such that*

W_{RB} *is the (non-empty) set of maximal* **RB**-*consistent sets.*

$\Gamma \; R_{RB}\mathcal{G} \;\Leftrightarrow\; \forall \Box\varphi \in \Gamma \; \exists \Delta \in \mathcal{G} : \varphi \in \Delta$ *for all* $\Gamma \in W_{RB}$, *and* $\mathcal{G} \subseteq W_{RB}$.

$V_{RB}(\Gamma, p) = 1 \Leftrightarrow p \in \Gamma$ *for all* $\Gamma \in W_{RB}$, *and* $p \in I\!P$.

Theorem 1

For any maximal **RB**-*consistent set* Γ *and for all* $\varphi \in \mathcal{L}^\Box$:

$$M_{RB}, \Gamma \models \varphi \Leftrightarrow \varphi \in \Gamma$$

The proof of this Henkin-style theorem leans on two lemmas. The first lemma, known as the Lindenbaum-lemma, states that every consistent set has a maximal consistent extension. This claim holds for **RB** because it contains the classical propositional calculus [HuC].

Lemma 1 *If Λ is an* **RB**-*consistent set, then there exists a maximal* **RB**-*consistent extension* Γ: $\Lambda \subseteq \Gamma$.

Lemma 2 *If $\Box\alpha \wedge \neg\Box\beta$ is* **RB**-*consistent, then also $\alpha \wedge \neg\beta$ is.*[13]

proof

Suppose $\alpha \wedge \neg\beta$ is **RB**-inconsistent.

$$\vdash_{RB} \neg(\alpha\wedge\neg\beta) \Rightarrow_{PC} \vdash_{RB} \alpha \rightarrow \beta \Rightarrow_{R3} \vdash_{RB} \Box\alpha \rightarrow \Box\beta \Rightarrow_{PC} \vdash_{RB} \neg(\Box\alpha\wedge\neg\Box\beta)$$

And so, $\Box\alpha \wedge \neg\Box\beta$ is also **RB**-inconsistent. ✐

proof of theorem 1.

By structural induction on the construction of formulae. The theorem is by definition of V_{RB} valid for primitive propositions. The induction-steps for $\neg$ and $\wedge$ are immediately derived from the induction-hypothesis. The only step that needs some clarification is the $\Box$-step.

Let $\Box\varphi \in \Gamma$. Clearly we obtain by the definition of R_{RB}, that for all $\mathcal{G}$ such that $\Gamma R_{RB}\mathcal{G}$ there exists $\Delta \in \mathcal{G}$ such that $\varphi \in \Delta$. By the induction-hypothesis we conclude that for all such Δ: $M_{RB}, \Delta \models \varphi$. From this conclusion we obtain directly $M_{RB}, \Gamma \models \Box\varphi$

Let $\Box\varphi \notin \Gamma$. Let $\{\alpha_i\}_{i \in I \subseteq \mathbb{N}}$ be an enumeration of the set $\{\alpha \mid \Box\alpha \in \Gamma\}$. Because $\neg\Box\varphi \in \Gamma$ we have **RB**-consistency of $\Box\alpha_i \wedge \neg\Box\varphi$ for every $i \in I$. By lemma 2 we conclude that $\alpha_i \wedge \neg\varphi$ is **RB**-consistent for every $i \in I$. Let $\mathcal{G} := \{\Gamma_i\}_{i \in I}$ be a set of maximal **RB**-consistent sets such that Γ_i extends $\{\alpha_i, \neg\varphi\}$ for all $i \in I$. Clearly, $\Gamma \; R_{RB}\mathcal{G}$ because for all $\Box\alpha \in \Gamma$ there exists a certain $\Gamma_i \in \mathcal{G}$ such that $\alpha \in \Gamma_i$. Because $\neg\varphi \in \Gamma_i$ for all $\Gamma_i \in \mathcal{G}$, and so by observation 2 $\varphi \notin \Gamma_i$ for all $\Gamma_i \in \mathcal{G}$, we infer from the induction-hypothesis $M_{RB}, \Gamma_i \not\models \varphi$ for all $\Gamma_i \in \mathcal{G}$, and so $M_{RB}, \Gamma \not\models \Box\varphi$. ✐

corollary **RB** is complete w.r.t. the class of fused Kripke-models.

proof

Let $\Phi \not\vdash_{RB} \varphi$. This causes $\Phi \cup \{\neg\varphi\}$ to be **RB**-consistent. Let Γ be a maximal **RB**-consistent extension of this set. By theorem 1 we conclude immediately $M_{RB}, \Gamma \models \alpha$ for all $\alpha \in \Phi$ and $M_{RB}, \Gamma \not\models \varphi$, and so $\Phi \not\models \varphi$. ✐

[13] In classical modal logic we have for any set of formulae Λ if $\Lambda \cup \{\neg\Box\alpha\}$ is **K**-consistent, then $\Box^-\Lambda \cup \{\neg\alpha\}$ ($= \{\varphi \mid \Box\varphi \in \Lambda\} \cup \{\neg\alpha\}$) is **K**-consistent. This weakening in fused modal logic is completely compensated by the more liberal accessibility relation for reaching completeness.

2.2 Full and slim models

The following observations show that we can freely limit the accessibility-relation somewhat by the ordinary subset-relation. We may restrict the large class of Kripke-models to what we call full , or slim, models.

Definition 7

Let $F = \langle W, R\rangle$ be a fused Kripke-frame. The slimmed frame of F is the fused Kripke-frame $F^{\downarrow} = \langle W, R^{\downarrow}\rangle$ such that

$$xR^{\downarrow}Y \Leftrightarrow xRY \ \&\forall Y' \subset Y : \text{not } xRY'$$

The filled frame of F is the fused Kripke frame $F^{\uparrow} = \langle W, R^{\uparrow}\rangle$ such that

$$xR^{\uparrow}Y \Leftrightarrow xRY' \text{ for certain } Y' \subseteq Y.$$

If $M = \langle W, R, V\rangle$ is a fused Kripke model, we say that $M^{\downarrow} = \langle W, R^{\downarrow}, V\rangle$ is the slimmed model of M, and $M^{\uparrow} = \langle W, R^{\uparrow}, V\rangle$ is the filled model of M.

A model M (or frame F) is said to be slim iff $M = M^{\downarrow}$ ($F = F^{\downarrow}$). A model M (or frame F) is said to be full iff $M = M^{\uparrow}$ ($F = F^{\uparrow}$).

Observation 3

The canonical model M_{RB} is full.

Theorem 2

Let $M = \langle W, R, V\rangle$ be a fused Kripke model. For all $w \in W$: $M, w \models \varphi \Leftrightarrow M^{\downarrow}, w \models \varphi \Leftrightarrow M^{\uparrow}, w \models \varphi$ for all formulae $\varphi \in \mathcal{L}^{\Box}$.

proof

Easy by induction on the construction of formulae. ✐

corollary **RB** is sound and complete with respect to the class of slim fused Kripke models, and also with respect to the class of full fused Kripke models.

It turns out that the class of full models is easier to handle for giving semantical characterizations of systems that extend **RB**. The only logics that we will be dealing with here are logics for belief.

3 Confused belief

Definition 8

The logic of confused belief,[14] *denoted by* **CB**, *is the logic* **RB** *with the following two axioms added:*

$$\vdash_{CB} \neg\Box\bot$$

$$\vdash_{CB} \Box\varphi \rightarrow \Box\Box\varphi$$

The first axiom tells us that the agent never believes the absurd proposition $\bot$. The second axiom states the so-called *positive introspection* of an agent. Whenever an agent believes a proposition, he always believes that he believes it.

These axioms are from modal logic known as **Ver** and **4** respectively. These axioms were accepted by Hintikka [Hina] in his axiomatization of belief, **D4** (= **K** + **Ver** + **4**)[15]. The only thing that has changed here with respect to Hintikka's axioms is that **K** (logical omniscience) has been cancelled. Characterizing these axioms semantically, we end up with a similar class of models as for **D4** in normal modal logic [HuC]. **D4** is characterized by the class of serial transitive normal Kripke models. Reshaping seriality in a fusion-like way does not give any complication. Accommodating a new concept of transitivity in this setting is somewhat nastier. The following notation helps.

Let $M = \langle W, R, V\rangle$ be a model:

$$X\overline{R}Y \Leftrightarrow \forall x \in X : xR^{\uparrow}Y$$

$$X\underline{R}Y \Leftrightarrow \exists x \in X : xR^{\uparrow}Y$$

In the first case all members of a set of worlds X have a successor-set in Y, in the second case there exists such a world in X with a successor-set in Y.

Definition 9

A fused model $M = \langle W, R, V\rangle$ is said to be serial if it has no dead-ends:

$$\forall x \exists Y : xRY \ (xR^{\uparrow}W).$$

A fused model $M = \langle W, R, V\rangle$ is said to be F-transitive if

$$\forall x, Y, Z : xR^{\uparrow}Y \ \& \ Y\overline{R}Z \Rightarrow xR^{\uparrow}Z$$

F-transitivity of a fused frame says that for every world x that has a successor-set, in which all elements have a successor-set in a certain set of worlds Z, this world x must also have a successor-set in Z.

Theorem 3

CB *is characterized by the class of serial F-transitive models.*

[14] A better name is perhaps 'possibly confused belief'.

[15] **D** originally stands for the axiom $\Box\varphi \rightarrow \Diamond\varphi$. For normal modal logics (**K** containing) **D** and **Ver** coincide. **RB** + **D** does obviously not coincide with **RB** + **Ver**. Note that **D** 'normalizes **RB** : **K** + **D** = **RB** + **D**.

proof

We will leave seriality for **RB** + **Ver**, because it is derived similarly to the normal case **K** + **Ver**. We will only show here that **RB** + **4** is characterized by the class of F-transitive models.

Suppose that $\langle W, R\rangle$ is not F-transitive. This means there exists a world $w \in W$ and two subsets $Y, Z \subseteq W$ such that $xR^{\uparrow}Y$ and $Y\overline{R}Z$ and not $xR^{\uparrow}Z$ (and so $Y \not\subseteq Z$). Let $\langle W, R, V\rangle$ be a model on this frame such that $V(z,p) = 0$ for all $z \in Z$ and $V(w,p) = 1$ for all $w \in W/Z$. Because x has no successor-set in Z, we conclude $M, x \models \Box p$. Furthermore because every $y \in Y$ does have such a successor-set in Z, we conclude $M, y \not\models \Box p$ for all $y \in Y$. This forces $M, x \not\models \Box\Box p$.

Conversely, suppose that $\langle W, R\rangle$ is F-transitive. Let M be an arbitrary model on $\langle W, R\rangle$ and let x be a world in W. Suppose $M, x \not\models \Box\Box p$. This means there exists $Y \subseteq W$ such that xRY and for all $y \in Y : M, y \not\models \Box p$, and so for all these y there exists Z_y such that yRZ_y and for all $z \in Z_y : M, z \not\models p$. We define

$$Z := \bigcup_{y \in Y} Z_y$$

Clearly, $xR^{\uparrow}Y$ and $Y\overline{R}Z$. By the F-transitivity we conclude $xR^{\uparrow}Z$. This gives us $M, x \not\models \Box p$. ✐

Theorem 4

CB *is sound and complete w.r.t. the class of serial F-transitive models.*

proof

We show that the **CB** canonical model $M_{CB} = \langle W_{CB}, R_{CB}, V_{CB}\rangle$ [16] is F-transitive. Its seriality is immediately obtained by lemma 2, because maximal **CB**-consistent sets always contain at least one $\neg\Box$-sentence. So, for every maximal **CB**-consistent set there is at least one successor-set (see proof of th. 1).

Let $\Gamma R_{CB} \mathcal{G}$ and $\mathcal{G}\overline{R}_{CB}\mathcal{G}'$. Let $\Box\varphi$ be an arbitrary belief-sentence in Γ. Clearly also $\Box\Box\varphi \in \Gamma$. By the two relational claims above we learn that $\varphi \in \cup\mathcal{G}'$[17], and so $\Gamma R_{CB}\mathcal{G}'$ (remember $R_{CB} = R^{\uparrow}_{CB}$). ✐

3.1 Full introspective confused belief

Sometimes the axiom of *negative introspection* is accepted in doxastic logic (e.g. [FaV] [Moo]). This axiom states that if an agent disbelieves a proposition, he believes that he does not believe it.

$$\vdash \neg\Box\varphi \rightarrow \Box\neg\Box\varphi$$

[16] This is the model M_{RB} restricted to the maximal **CB**-consistent sets.

[17] $\cup\mathcal{G} = \bigcup_{\Gamma \in \mathcal{G}} \Gamma$. Note that $\Gamma R_{RB}\mathcal{G} \Leftrightarrow \Box^{-}\Gamma \subseteq \cup\mathcal{G}$.

This axiom is known as **5** in classical modal logic. **D45** is characterized by the serial transitive Euclidean normal Kripke frames. Euclidean relations are those in which any pair of successors of a given world are mutually related:

$$\forall x, y, z : xRy \,\&\, xRz \Rightarrow yRz$$

If we characterize **RB + Ver + 4 + 5** we find the following transformation of Euclidicity:

$$\forall x, Y, Z : xR^{\uparrow}Y \,\&\, xR^{\uparrow}Z \Rightarrow Y\underline{R}Z$$

This F-Euclidicity states that each member of a pair of successor-sets of a world has at least one element that has a successor-set in the other member of this pair. Again we have found a close correspondence with the classical Kripke semantics. The funny thing here is that we need to use the existential $\underline{R}$-relation, while for transitivity we needed the universal $\overline{R}$. This is caused by the 'double' duality between $\Diamond$ and $\Box$. $\Diamond\varphi$ is verified in a world whenever there exists a successor-set of which all members verify φ.

4 Conclusions

We showed that it is possible to represent inconsistent beliefs in a possible world framework. We still do not accept that an agent may believe the absurd proposition $\perp$, in contrast to the four valued approach in which $\perp$ is taken to be a meaningful proposition. The semantics is intuitively appealing and we stay as close as possible to ordinary Kripke- or possible worlds semantics. The Henkin style completeness proof indicates that we have a modal logic that is *abnormal*, but with a regular character. Also correspondence results gave evidence for this regularity. [18] Other well known axioms such as $\mathbf{T} = \Box\varphi \rightarrow \varphi$ and $\mathbf{B} = \varphi \rightarrow \Box\Diamond\varphi$ find similar characterizations. The first system imposes reflexivity on normal Kripke frames. **RB + T** gets characterized by the fused frames such that all non-empty subsets of the set of worlds contains an element that has a successor-set in this set itself. Formally we obtain the following fusion-style lookalike: $\forall X : X\underline{R}X$. The system **B** defines normally the class of symmetric frames. **RB + B** also ends up with a fusion-style version of symmetry: $\forall X, Y : X\overline{R}Y \Rightarrow Y\underline{R}X$; every non-empty set of which all the elements have a successor-set in some other set, is a superset of some successor-set of one of the elements of this other set. This relatively nice behavior gives us a comfortable position between the logics that deal with inconsistent belief. Lots of the meta-theory of classical modal logic can be adopted [Jas]

Only a part of logical omniscience has been dropped by skipping to fused modal logic for reasoning about belief. Necessitation, that is believing all tautologies, is still there. We think that if we have to overcome this problem, we have to incorporate fusion into partial modal logic (see [Thi]), in which total worlds are replaced by situations or partial worlds. This challenge of uniting these two approaches really offers an alternative for four valued logic, and it will be the main theme of further research. [19]

---

[18] Its regularity is also demonstrated by other inherited analogues, such as preservation-results [Jas]. E.g. by appropriate modification we have closure under generated submodelling, disjoint unions and p-morphisms (compare [Ben] [JoV] [Gol]).

[19] We think that four valued logic has also come up by its nice symmetric Boolean behavior [Lan] [Mus],. It seems to us that four valued logic has become popular by the natural attractiveness of partial semantics on the one hand, and on the other hand a kind of homesickness to two-valued logic of logicians who went partial. ***Undefined*** is incorporated in the set of truth-values (with equal status) and so there is a natural – from a two-valued viewpoint – quest for a Boolean complementary truth-value ***overdefined***.

References

[Bel] N.D. Belnap, "A useful four-valued logic," in *Modern Uses of Multiple Valued Logic*, G. Epstein & M. Dunn, eds., Reidel, Dordrecht, 1977.

[Ben] J.F.A.K. van Benthem, *Modal Logic and Classical Logic*, Bibliopolis, Naples, 1985.

[FaV] R. Fagin & M.Y. Vardi, *An Internal Semantics for Modal Logic: Preliminary Report*, CSLI Research Notes #85-25, Stanford University, Stanford, 1985.

[Gol] R. Goldblatt, *Logics of Time and Computation*, CSLI Lecture Notes #7, Stanford University, Stanford, 1987.

[Gri] H.P. Grice, "Logic and conversation," in *Speech Acts, Syntax and Semantics*, P.Cole & J. Morgan, eds. #III: Speech Acts, Academic Press, New York, 1975.

[Hina] J. Hintikka, *Knowledge and Belief: An Introduction to the Logic of the Two Notions*, Cornell University Press, Ithaca N.Y., 1962.

[Hinb] J. Hintikka, "Impossible possible worlds vindicated," *Journal of Philosophical Logic* 4 (1975), 475–484.

[HuC] G.E. Hughes & M.J. Cresswell, *A Companion to Modal Logic*, Methuen, New York, 1984.

[Jas] J.O.M. Jaspars, "Fused modal logic and inconsistent belief," in *Proceedings of the First World Conference on the Fundamentals of AI*, M. de Glas & D.M. Gabbay, eds., Angkor Publishing Company, Paris, 1991 .

[JoV] D. de Jongh & F. Veltman, "Intensionele Logica," Lecture Notes on Intensional Logic, University of Amsterdam, 1985.

[Lan] T. Langholm, *Partiality, Truth and Persistence*, CSLI Lecture Notes #15, Stanford University, Stanford, 1988.

[Lev] H.J. Levesque, "A logic of implicit and explicit belief," in *Proceedings of Sixth Conference of the American Association of Artificial Intelligence*, Morgan Kaufmann, Los Altos, 1984, 198–202.

[Moo] R.C. Moore, "Possible world semantics for autoepistemic logic," in *Proceedings of the AAAI Workshop on Non-Monotonic Reasoning*, New Paltz, NY, 1984, 344–354.

[Mus] R.A. Muskens, "Meaning and Partiality," University of Amsterdam, PhD.-thesis, 1989.

[Ran] V. Rantala, "Impossible world semantics and logical omniscience," *Acta Philosophica Fennica* 35 (1982), 106–115.

[ReB] N. Rescher & R. Brandom, *The Logic of Inconsistency*, Basil Blackwell, London, 1980.

[Thi] E.G.C. Thijsse, "Partial propositional and modal logic; the overall theory," in *Proceedings of the Seventh Amsterdam Colloquium*, M. Stokhof & L. Torenvliet, eds., ITLI, Amsterdam, 1990.